SCHUBERT VARIETIES AND THE FUSION PRODUCTS. THE GENERAL CASE.

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Abstract. This paper generalizes the results of the paper [FF3] to the case of the general $\mathfrak{sl}_2$ Schubert varieties. We study the homomorphisms between different Schubert varieties, describe their geometry and the group of the line bundles. We also derive some consequences concerning the infinite-dimensional generalized affine grassmanians.

Introduction

Let $A=(a_1,\dots,a_n)\in\mathbb{N}^n, 1<a_1\le\dots\le a_n$. Let M^A be the fusion product of $\mathfrak{sl}_2$ modules $\mathbb{C}^{a_1},\dots,\mathbb{C}^{a_n}$ (see [FL1, FF1]) (recall that M^A is $\mathfrak{sl}_2\otimes(\mathbb{C}[t]/t^n)$ module). Let v_A be the lowest weight vector in M^A with respect to the h_0-grading (for $x\in\mathfrak{sl}_2$ denote $x_i=x\otimes t^i$). Denote by $[v_A]$ the line $\mathbb{C}\cdot[v_A]\in\mathbb{P}(M^A)$. In [FF3] the closure $\mathrm{sh}_A=\overline{\mathrm{SL}_2(\mathbb{C}[t]/t^n)\cdot[v_A]}\hookrightarrow\mathbb{P}(M^A)$ was studied in the case when $a_i\ne a_j$ for $i\ne j$. It was proved that in this case the Schubert variety sh_A is smooth n-dimensional algebraic variety, independent on the choice of A (the only demand is $a_i\ne a_j$). This variety was denoted by $\mathrm{sh}^{(n)}$. In the present paper we study the case of the general Schubert variety.

We start with the generalization of the independence of sh_A $(a_i\ne a_j)$ on A. We say that A is of the type $\{i_1,\dots,i_s\}$ if $i_1+\dots+i_s=n$ and

$$a_1=\dots=a_{i_1}\ne a_{i_1+1}=\dots=a_{i_1+i_2}\ne\dots\ne a_{i_1+\dots+i_{s-1}+1}=\dots=a_n.$$

Then $\mathrm{sh}_A\simeq\mathrm{sh}_B$ if and only if A and B are of the same type. We denote the corresponding variety by $\mathrm{sh}_{\{i_1,\dots,i_s\}}$ and the point $[v_A]$ by $[v_{\{i_1,\dots,i_s\}}]$. For example, $\mathrm{sh}^{(n)}=\mathrm{sh}_{\{1,\dots,1\}}$.

In [FF3] for any $n>k$ the bundle $\mathrm{sh}^{(n)}\to\mathrm{sh}^{(k)}$ with a fiber $\mathrm{sh}^{(n-k)}$ was constructed. The generalization of this construction is the following fact: let $1\le t<s$. Then there exists $\mathrm{SL}_2(\mathbb{C}[t]/t^n)$-equivariant bundle $\mathrm{sh}_{\{i_1,\dots,i_s\}}\to\mathrm{sh}_{\{i_{t+1},\dots,i_s\}}$ with a fiber $\mathrm{sh}_{\{i_1,\dots,i_t\}}$, sending $[v_{\{i_1,\dots,i_s\}}]$ to $[v_{\{i_{t+1},\dots,i_s\}}]$. For the proof we study some special subvarieties of $\mathrm{sh}_{\{i_1,\dots,i_s\}}$. Namely, we prove that

$$\mathrm{sh}_{\{i_1,\dots,i_s\}}=\mathrm{SL}_2(\mathbb{C}[t]/t^n)\cdot[v_{\{i_1,\dots,i_s\}}]\cup\bigcup_{j=1}^{s-1}N_{i_1+\dots+i_j}(\{i_1,\dots,i_s\}),$$

where $N_\alpha(\{i_1,\dots,i_s\})$ are some subvarieties of $\mathrm{sh}_{\{i_1,\dots,i_s\}}$. The latter are closely connected with $\mathfrak{sl}_2\otimes(\mathbb{C}[t]/t^n)$ submodules $S_{i,i+1}(A)\hookrightarrow M^A$, studied in [FF2, FF3]. Recall that these submodules were defined via the following exact sequence of $\mathfrak{sl}_2\otimes(\mathbb{C}[t]/t^n)$ modules (see [SW1, SW2] for the similar relations on the q-characters):

$$0\to S_{i,i+1}(A)\to M^A\to M^{(a_1,\dots,a_i-1,a_{i+1}+1,\dots,a_n)}\to 0.$$

Now let

$$A_{\{i_1,\dots,i_s\}} = (2^{i_1}\dots(s+1)^{i_s}) = (\underbrace{2,\dots,2}_{i_1},\dots,\underbrace{(s+1),\dots,(s+1)}_{i_s}).$$

Fix an isomorphism $\mathrm{sh}_{A_{\{i_1,\dots,i_s\}}} \simeq \mathrm{sh}_{\{i_1,\dots,i_s\}}$. Then

$$N_\alpha(\{i_1,\dots,i_s\}) \hookrightarrow \mathbb{P}(S_{\alpha,\alpha+1}(A_{\{i_1,\dots,i_s\}})).$$

An important property is the existence of the surjective homomorphisms between the different Schubert varieties. Namely, we say that $\{i_1,\dots,i_s\} \ge \{j_1,\dots,j_{s_1}\}$ if there exists such numbers $1 \le k_1 < \dots < k_{s_1-1} < s$ that

$$i_1+\dots+i_{k_1} = j_1,\dots,i_{k_{s_1-1}+1}+\dots+i_s = j_{s_1}.$$

For example, $\{1,\dots,1\}$ is the largest type, while $\{n\}$ is the smallest one. We will prove that if $\{i_1,\dots,i_s\} \ge \{j_1,\dots,j_{s_1}\}$ then there exists a surjective $\mathrm{SL}_2(\mathbb{C}[t]/t^n)$-equivariant homomorphism $\mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{j_1,\dots,j_{s_1}\}}$.

Schubert varieties have a description in terms of some partial flags in $W_0 = \mathbb{C}^2 \otimes \mathbb{C}[t]$ (for the analogous construction in the case of $\mathrm{sh}^{(n)}$ see [FF3]; see also [SP]). W_0 is naturally $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ module and we also have an action of the operator t by multiplication. Consider the variety $\mathrm{Fl}_{\{i_1,\dots,i_s\}}$ of the sequences of the subspaces of W_0:

$$\begin{aligned}\mathrm{Fl}_{\{i_1,\dots,i_s\}} = \{&W_0 \hookleftarrow W_1 \hookleftarrow \dots \hookleftarrow W_s : \\ &1).\ tW_\alpha \hookrightarrow W_\alpha;\quad 2).\ \dim W_\alpha/W_{\alpha+1} = i_{s-\alpha};\quad 3).\ W_{\alpha+1} \hookleftarrow t^{i_{s-\alpha}}W_\alpha\}.\end{aligned}$$

We prove that $\mathrm{Fl}_{\{i_1,\dots,i_s\}} \simeq \mathrm{sh}_{\{i_1,\dots,i_s\}}$.

We study the Picard group of the Schubert varieties. Let $\mathcal{E}$ be the line bundle on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$. Following [FF3] denote by C_i the projective lines:

$$C_i = \overline{\{\exp(ze_i)\cdot[v_{\{i_1,\dots,i_s\}}], z \in \mathbb{C}\}} \hookrightarrow \mathrm{sh}_{\{i_1,\dots,i_s\}}, i = 0,\dots,n-1.$$

One can show that $\mathcal{E}$ is completely determined by its restriction to these lines. Let $B = (b_1,\dots,b_n) \in \mathbb{Z}^n$. Introduce a notation $\mathcal{E} = \mathcal{O}(B)$ if $\mathcal{E}|_{C_i} \simeq \mathcal{O}(b_1+\dots+b_{n-i})$. We prove that the bundle $\mathcal{O}(B)$ really exists on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$ if and only if the type of B is less or equal to $\{i_1,\dots,i_s\}$. It is possible to describe the space of sections of $\mathcal{O}(B)$ in terms of the fusion products. Namely if $b_i \ge 0$ then

$$H^0(\mathcal{O}(B), \mathrm{sh}_{\{i_1,\dots,i_s\}}) \simeq \left(M^{(b_1+1,\dots,b_n+1)}\right)^*.$$

As a consequence we obtain the proof of the theorem about the sections of the line bundles on the generalized affine grassmanians (see [FF3, FS, FL2]). Recall the main definitions. An embedding $\mathrm{sh}_{\{i_1,\dots,i_s\}} \hookrightarrow \mathrm{sh}_{\{i_1,\dots,i_s+2\}}$ allows us to organize an inductive limit

$$Gr_{\{i_1,\dots,i_s\}} = \lim_{k\to\infty} \mathrm{sh}_{(i_1,\dots,i_s+2k)}.$$

There exists a bundle $\mathcal{O}(B^{(\infty)})$ on $Gr_{\{i_1,\dots,i_s\}}$ such that

$$\mathcal{O}(B^{(\infty)})|_{\mathrm{sh}_{\{i_1,\dots,i_s+2k\}}} = \mathcal{O}(b_1,\dots,b_n,\underbrace{b_n,\dots,b_n}_{2k}).$$

We decompose the space of sections of the above bundle as $\widehat{\mathfrak{sl}_2}$ module in the case $0 \le b_1 \le \ldots \le b_n$:

$$H^0(\mathcal{O}(B^{(\infty)}), Gr_{\{i_1,\ldots,i_s\}}) \simeq \bigoplus_{j=0}^{b_n} c_{j;b_1,\ldots,b_n} L^*_{j,b_n}.$$

Here L_{j,b_n} are irreducible $\widehat{\mathfrak{sl}_2}$ modules and $c_{j;b_1,\ldots,b_n}$ – the structure constants of the level b_n+1 Verlinde algebra, associated with the Lie algebra $\mathfrak{sl}_2$.

We finish the paper with the discussion of the singularities of the Schubert varieties. Namely we prove that the only smooth Schubert variety is $\mathrm{sh}^{(n)}$ and study the singularities of the "smallest" variety $\mathrm{sh}_{\{n\}}$.

The paper is organized in the following way:

Section 1 contains the preliminary statements from the papers [FF1, FF2, FF3] and some generalizations (lemma (1.1)).

In the section 2 we identify the isomorphic Schubert varieties (theorem (2.1)).

Section 3 is devoted to the proof of the existence of the bundle $\mathrm{sh}_{\{i_1,\ldots,i_s\}} \to \mathrm{sh}_{\{i_{t+1},\ldots,i_s\}}$ with a fiber $\mathrm{sh}_{\{i_1,\ldots,i_t\}}$ (theorem (3.1)).

In the section 4 we give the description of $\mathrm{sh}_{\{i_1,\ldots,i_s\}}$ in terms of the generalized partial flag manifolds (proposition (4.1)).

Section 5 is devoted to the study of the line bundles on $\mathrm{sh}_{\{i_1,\ldots,i_s\}}$ (proposition (5.1)) and the spaces of their sections (corollary (5.1)).

In the section 6 we decompose the spaces of sections of some line bundles on the infinite-dimensional generalized affine grassmanians into the sum of the dual irreducible $\widehat{\mathfrak{sl}_2}$ modules (proposition (6.1)).

Section 7 contains the discussion of the singularities of the Schubert varieties.

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1. Preliminaries

Here we briefly recall the main notions and statements about the Schubert varieties sh_A from [FF3].

Let $A = (a_1, \ldots, a_n) \in \mathbb{N}^n, 1 < a_1 \le \ldots \le a_n$, M^A the corresponding fusion product (see [FL1, FF1]), v_A and u_A its lowest and highest weight vectors with respect to the h_0-grading (for $x \in \mathfrak{sl}_2$ $x_i = x \otimes t^i$). M^A is cyclic $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^n)$ module, $M^A = \mathbb{C}[e_0, \ldots, e_{n-1}] \cdot v_A$. The group $\mathrm{G} = \mathrm{SL}_2(\mathbb{C}[t]/t^n)$ acts on M^A and thus on its projectivization $\mathbb{P}(M^A)$. Schubert variety $\mathrm{sh}_A \hookrightarrow \mathbb{P}(M^A)$ is the closure of the orbit of the point $[v_A]$ (for $v \in M^A$ $[v] = \mathbb{C} \cdot v \in \mathbb{P}(M^A)$):

$$\mathrm{sh}_A = \overline{\mathrm{G} \cdot [v_A]}.$$

It was proved in [FF3] that sh_A is projective complex algebraic variety. Its coordinate ring can be described in the following way. Recall that for $A, B, C \in \mathbb{N}^n$ with $c_i = a_i + b_i - 1$ the multiplication $(M^A)^* \otimes (M^B)^* \to (M^C)^*$ was constructed. Thus, for any A we have an algebra

$$CR_A = \bigoplus_{i=0}^{\infty} (M^{A_i})^*,$$

where $A_i = (ia_1 - i + 1, \ldots, ia_n - i + 1),\ i \ge 0$. It was proved in [FF3] that CR_A is a coordinate ring of sh_A.

We will need the realization of the fusion products in the tensor powers of the space of the semi-infinite forms (fermionic realization), constructed in [FF2]. Recall that the space of the semi-infinite forms F (the fermionic space) is the level 1 $\widehat{\mathfrak{sl}_2}$ module. F contains the set of vectors $v(i), i \in \mathbb{Z}$ with the following properties:

$$1.\ \ e_{i-1}v(i) = v(i-2), \qquad 2.\ \ e_k v(i) = 0 \text{ if } k \geq i,$$

$$3.\ \ \mathrm{U}(\widehat{\mathfrak{sl}_2}) \cdot v(0) \simeq L_{0,1}, \qquad 4.\ \ \mathrm{U}(\widehat{\mathfrak{sl}_2}) \cdot v(1) \simeq L_{1,1}.$$

It was proved in [FF2] that M^A can be embedded into $F^{\otimes(a_n-1)}$. Namely,

$$M^A \simeq \mathrm{U}(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) \cdot (v(n) \otimes v(n-d_2) \otimes \ldots \otimes v(n-d_2-\ldots-d_{a_n-1})) \,, \tag{1}$$

where $d_i = \#\{\alpha : \ a_\alpha = i\}$.

Recall the special submodules of M^A, studied in [FF3]. For any $i, 1 \leq i < n$ where exists an $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ submodule $S_{i,i+1}(A) \hookrightarrow M^A$ with a property

$$M^A / S_{i,i+1}(A) \simeq M^{(a_1,\ldots,a_{i-1},a_i-1,a_{i+1}+1,a_{i+2},\ldots,a_n)}.$$

There are three cases when the modules $S_{i,i+1}(A)$ can be easily described:

$$1.\ \ S_{1,2}(A) \simeq M^{(a_2-a_1+1,a_3,\ldots,a_n)};$$

$$2.\ \ \text{if } a_i = a_{i+1}, \text{ then } S_{i,i+1}(A) \simeq M^{(a_1,\ldots,a_{i-1},a_{i+2},\ldots,a_n)};$$

$$3.\ \ S_{n-1,n}(A) \simeq M^{(a_1,\ldots,a_{n-2})} \otimes \mathbb{C}^{a_n-a_{n-1}+1}.$$

In the general case we have the following lemma, which follows from the fermionic realization of the fusion product (for the analogous proofs see [FF3]).

Lemma 1.1. *Let $a_i < a_{i+1}$. Denote*

$$A' = (a_1, \ldots, a_{i-1}, a_{i+2}, \ldots, a_n), \qquad A'' = (a_{i+1} - a_i + 1, \ldots, a_n - a_i + 1).$$

Then there is an embedding of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ modules

$$S_{i,i+1}(A) \hookrightarrow M^{A'} \otimes M^{A''}.$$

The image of this embedding coincides with

$$\mathbb{C}[e_0, \ldots, e_{n-1}, e^{(2)}_{n-i-1}] \cdot (v_{A'} \otimes v_{A''}) \,, \tag{2}$$

where $e^{(i)}_j$ stands for the operator e_j acting on the i-th factor of the tensor product. From now on we identify $S_{i,i+1}(A)$ with its image (2). The vector $v_{A'} \otimes v_{A''}$ (as well as its preimage in $S_{i,i+1}(A)$) is denoted by $v_{i,i+1}(A)$.

2. The main definition

Let $A = (a_1 \leq \ldots \leq a_n) \in \mathbb{Z}^n$. We say that A is of the type $\{i_1, \ldots, i_s\}$ with $i_1 + \ldots + i_s = n, i_\alpha \geq 1$ if

$$a_1 = \ldots = a_{i_1} \neq a_{i_1+1} = a_{i_1+2=} \ldots = a_{i_1+i_2} \neq \ldots \neq a_{n-i_s+1} = \ldots = a_n.$$

Now let $A, B \in (\mathbb{N} \setminus 1)^n$. We will prove that if A and B are of the same type, then $\mathrm{sh}_A \simeq \mathrm{sh}_B$. Introduce a notation: if A is of the type $\{i_1, \ldots, i_s\}$, then we denote

$$A = (a_1^{i_1} a_{i_1+1}^{i_2} \ldots a_{i_1+\ldots+i_{s-1}+1}^{i_s}).$$

Lemma 2.1. $\mathrm{sh}_{(2^n)} \simeq \mathrm{sh}_{(k^n)}$ *for any $k \geq 2$.*

Proof. Recall that the coordinate ring of $\mathrm{sh}_{(2^n)}$ is $\bigoplus_{i=0}^\infty (M^{(i^n)})^*$. The coordinate ring of $\mathrm{sh}_{(k^n)}$ is $\bigoplus_{j=0}^\infty (M^{((jk-j+1)^n)})^*$. Thus $\mathrm{sh}_{(2^n)} \simeq \mathrm{sh}_{(k^n)}$ (see [H]). $\square$

Theorem 2.1. *Let $A, B \in (\mathbb{N}\setminus 1)^n$ be of the same type. Then* $\mathrm{sh}_A \simeq \mathrm{sh}_B$.

Proof. It is enough to show that if A is of the type $\{i_1,\dots,i_s\}$, then

$$\mathrm{sh}_A \simeq \mathrm{sh}_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}.$$

Note that there is a natural surjective G-equivariant homomorphism

$$\mathrm{sh}_A \to \mathrm{sh}_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}, \quad [v_A] \mapsto [v_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}].$$

For the proof we use the fermionic realization of the fusion product. Let ϕ_A be the embedding $M^A \hookrightarrow F^{\otimes(a_n-1)}$ (sometimes we write simply ϕ instead of ϕ_A). Then, because of (1)

$$(3)\quad \phi_A(v_A) = v(n)^{\otimes(a_1-1)}\otimes \\ \otimes v(n-i_1)^{\otimes(a_{i_1+1}-a_{i_1})}\otimes\dots\otimes v(n-i_1-\dots-i_{s-1})^{\otimes(a_{n-i_s+1}-a_{i_s})}.$$

Picking one factor from each tensor power

$$v(n-i_1-\dots-i_\alpha)^{\otimes(a_{i_1+\dots+i_\alpha+1}-a_{i_1+\dots+i_\alpha})}$$

we obtain a vector

$$\phi_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}(v_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}) \in F^{\otimes s}.$$

Thus for some permutation $\sigma \in S_{a_n-1}$ of the factors of $F^{\otimes(a_n-1)}$ and a vector $w \in F^{\otimes(a_n-s-1)}$ we have

$$\sigma\phi_A(v_A) = \phi_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}(v_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}) \otimes w.$$

Hence because of $\mathrm{sh}_A = \overline{\mathrm{G}\cdot[v_A]}$ we obtain a surjective map

$$\mathrm{sh}_A \to \mathrm{sh}_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}, \quad [v_A] \mapsto [v_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})}].$$

We need to prove that this map is an isomorphism.

Note that from (3) we obtain an embedding

$$\mathrm{sh}_A \hookrightarrow \mathrm{sh}_{((a_1)^n)} \times \mathrm{sh}_{((a_{i_1+1}-a_{i_1}+1)^{n-i_1})} \times \dots \times \mathrm{sh}_{((a_n-a_{n-i_s}+1)^{n-i_1-\dots-i_{s-1}})}.$$

Also we have an embedding

$$\mathrm{sh}_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})} \hookrightarrow \mathrm{sh}_{(2^n)} \times \mathrm{sh}_{(2^{n-i_1})} \times \dots \times \mathrm{sh}_{(2^{n-i_1-\dots-i_{s-1}})}$$

and the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{sh}_A \arrow[r] \arrow[d] & \mathrm{sh}_{((a_1)^n)} \times \dots \times \mathrm{sh}_{((a_n-a_{n-i_s}+1)^{n-i_1-\dots-i_{s-1}})} \arrow[d, "\psi"'] \\
\mathrm{sh}_{(2^{i_1}3^{i_2}\dots(s+1)^{i_s})} \arrow[r] & \mathrm{sh}_{(2^n)} \times \mathrm{sh}_{(2^{n-i_1})} \times \dots \times \mathrm{sh}_{(2^{n-i_1-\dots-i_{s-1}})}.
\end{tikzcd}$$

Note that because of the lemma (2.1), the map ψ is an isomorphism as the product of the isomorphisms

$$\mathrm{sh}_{((a_{i_1+\dots+i_\alpha+1}-a_{i_1+\dots+i_\alpha}+1)^{n-i_1-\dots-i_\alpha})} \simeq \mathrm{sh}_{(2^{n-i_1-\dots-i_\alpha})}.$$

Thus the left vertical map from the diagram is an isomorphism. $\square$

Definition 2.1. *Let $A \in (\mathbb{N}\setminus 1)^n$ be of the type $\{i_1,\dots,i_s\}$. Define* $\mathrm{sh}_{\{i_1,\dots,i_s\}} = \mathrm{sh}_A$. *Denote* $[v_A] = [v_{\{i_1,\dots,i_s\}}],\quad [u_A] = [u_{\{i_1,\dots,i_s\}}]$.

Example. Recall that in [FF3] the case of the sh_A with pairwise distinct a_i was considered. It was proved that all the Schubert varieties of this type are isomorphic. This variety was denoted as $\mathrm{sh}^{(n)}$. In our notations $\mathrm{sh}^{(n)} = \mathrm{sh}_{\underbrace{\{1,\dots,1\}}_{n}}$.

3. The existence of the bundle $\mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{i_{t+1},\dots,i_s\}}$

Recall that in [FF3] for any $n > k$ a bundle $\pi_{n,k} : \mathrm{sh}^{(n)} \to \mathrm{sh}^{(k)}$ with a fiber $\mathrm{sh}^{(n-k)}$ was constructed. In this section we prove that for any $t < s$ there exists G-equivariant bundle

$$\mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{i_{t+1},\dots,i_s\}} \quad \text{with a fiber} \quad \mathrm{sh}_{\{i_1,\dots,i_t\}}.$$

In order to do that, we study the structure of some special subvarieties of $\mathrm{sh}_{\{i_1,\dots,i_s\}}$.

Lemma 3.1. *Let $1 \le \alpha < s$. Then there is a G-equivariant surjective homomorphism*

$$\mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{i_1,\dots,i_{\alpha-1},i_\alpha+i_{\alpha+1},i_{\alpha+2},\dots,i_s\}}, \quad [v_{\{i_1,\dots,i_s\}}] \mapsto [v_{\{i_1,\dots,i_\alpha+i_{\alpha+1},\dots,i_s\}}],$$

and the restriction $G \cdot [v_{\{i_1,\dots,i_s\}}] \to \mathrm{G} \cdot [v_{\{i_1,\dots,i_\alpha+i_{\alpha+1},\dots,i_s\}}]$ is an isomorphism.

Proof. Fix the realizations

$$\mathrm{sh}_{\{i_1,\dots,i_s\}} = \mathrm{sh}_{(2^{i_1}\dots(s+1)^{i_s})}, \quad \mathrm{sh}_{\{i_1,\dots,i_\alpha+i_{\alpha+1},\dots,i_s\}} = \mathrm{sh}_{(2^{i_1}\dots(\alpha+1)^{i_\alpha+i_{\alpha+1}}\dots s^{i_s})}.$$

Note that

$$\begin{aligned} (4) \quad & \phi(v_{(2^{i_1}\dots(s+1)^{i_s})}) = v(n) \otimes v(n-i_1) \otimes \dots \otimes v(n-i_1-\dots-i_{s-1}); \\ & \phi(v_{(2^{i_1}\dots(\alpha+1)^{i_\alpha+i_{\alpha+1}}\dots s^{i_s})}) = v(n) \otimes \dots \otimes v(n-i_1-\dots-i_{\alpha-1}) \otimes \\ & \qquad \otimes v(n-i_1-\dots-i_\alpha-i_{\alpha+1}) \otimes \dots \otimes v(n-i_1-\dots-i_{s-1}). \end{aligned}$$

For the sequence $0 < n_1 < \dots < n_j \le n$ define a homomorphism of $\widehat{\mathfrak{sl}}_2$ modules $P(n_1,\dots,n_j) : F^{\otimes n} \to F^{\otimes j}$:

$$P(n_1,\dots,n_j)(w_1 \otimes \dots \otimes w_n) = w_{n_1} \otimes \dots \otimes w_{n_j}.$$

From the formula (4) we obtain

$$(5) \qquad P(1,2,\dots,\alpha,\alpha+2,\dots,s)\phi(v_{(2^{i_1}\dots(s+1)^{i_s})}) = \phi(v_{(2^{i_1}\dots(\alpha+1)^{i_\alpha+i_{\alpha+1}}\dots s^{i_s})}).$$

The formula (5) gives us the needed homomorphism. □

Introduce a notation: let $i_1+\dots+i_s = j_1+\dots+j_{s_1} = n$. We write $\{i_1,\dots,i_s\} \ge \{j_1,\dots,j_{s_1}\}$ if for any A, B of the types $\{i_1,\dots,i_s\}$, $\{j_1,\dots,j_{s_1}\}$ correspondingly the following condition holds: $(a_i = a_{i+1} \Rightarrow b_i = b_{i+1})$. For example, $\{1,\dots,1\}$ is a maximal type, while $\{n\}$ is a minimal one.

Corollary 3.1. *Let $\{i_1,\dots,i_s\} \ge \{j_1,\dots,j_{s_1}\}$. Then there exists a G-equivariant surjective homomorphism $\mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{j_1,\dots,j_{s_1}\}}$. In particular, for any type $\{i_1,\dots,i_s\}$ there exists a homomorphism $h_{\{i_1,\dots,i_s\}} : \mathrm{sh}^{(n)} \to \mathrm{sh}_{\{i_1,\dots,i_s\}}$.*

For the proof of the existence of the bundle $\mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{i_{t+1},\dots,i_s\}}$ we need to study the structure of the complement $\mathrm{sh}_{\{i_1,\dots,i_s\}} \setminus \mathrm{G} \cdot [v_{\{i_1,\dots,i_s\}}]$. First recall some results from [FF3]. Denote $B_n = (2,\dots,n+1)$. Thus B_n is of the type $\{1,\dots,1\}$ and $\mathrm{sh}_{B_n} = \mathrm{sh}^{(n)}$.

Statement 3.1. *There exist $(n-1)$-dimensional subvarieties $N_j(\{1,\dots,1\}) \hookrightarrow \mathrm{sh}^{(n)}$, $j=1,\dots,n-1$ with the following properties:*
1. $\mathrm{sh}^{(n)} \setminus \mathrm{G}\cdot[v_{B_n}] = \bigcup_{j=1}^{n-1} N_j(\{1,\dots,1\})$;
2. $N_j(\{1,\dots,1\}) \simeq \{(x,y)\in \mathrm{sh}^{(n-2)}\times \mathrm{sh}^{(n-j)} :\ \pi_{n-2,n-j-1}(x)=\pi_{n-j,n-j-1}(y)\}$;
3. $N_j(\{1,\dots,1\}) \hookrightarrow \mathbb{P}(S_{j,j+1}(B_n))$;
4. $N_j(\{1,\dots,1\}) = \overline{\{\exp\left(\sum_{i=0}^{n-1} e_i z_i + e^{(2)}_{n-j-1}z\right)\cdot[v_{j,j+1}(B_n)], z_i, z\in\mathbb{C}\}}$ *(the closure is taken in the projective space $\mathbb{P}(S_{j,j+1}(B_n))$; about the notation $e^{(2)}_{n-j-1}$ see lemma (1.1)).*

We will need the following lemma.

Lemma 3.2. *Denote $A_{\{i_1,\dots,i_s\}} = (2^{i_1}\dots(s+1)^{i_s})$. Then*

$$h_{\{i_1,\dots,i_s\}}[v_{j,j+1}(B_n)] = [v_{i_1+\dots+i_\alpha, i_1+\dots+i_\alpha+1}(A_{\{i_1,\dots,i_s\}})]$$

for all j with $i_1+\dots+i_{\alpha-1} < j \le i_1+\dots+i_\alpha$ (we put $i_0=0$).

Proof. Recall (see [FF3]) that

$$\phi_{B_n}(v_{j,j+1}(B_n)) = \\ = v(n-2)\otimes v(n-3)\otimes\dots\otimes v(n-j-1)\otimes v(n-j)\otimes v(n-j-1)\otimes\dots\otimes v(0).$$

At the same time

$$\phi_{A_{\{i_1,\dots,i_s\}}}(v_{i_1+\dots+i_\alpha,i_1+\dots+i_\alpha+1}(A_{\{i_1,\dots,i_s\}})) = v(n-2)\otimes v(n-i_1-2)\otimes\dots \\ \dots\otimes v(n-i_1-\dots-i_{\alpha-1}-2)\otimes v(n-i_1-\dots-i_\alpha)\otimes\dots\otimes v(n-i_1-\dots-i_{s-1}).$$

Thus for $i_1+\dots+i_{\alpha-1} < j \le i_1+\dots+i_\alpha$

$$P(1, i_1+1, i_1+i_2+1,\dots, i_1+\dots+i_{s-1}+1)(\phi_{B_n}(v_{j,j+1}(B_n))) = \\ = \phi_{A_{\{i_1,\dots,i_s\}}}(v_{i_1+\dots+i_\alpha,i_1+\dots+i_\alpha+1}(A_{\{i_1,\dots,i_s\}})).$$

This gives us $h_{\{i_1,\dots,i_s\}}[v_{j,j+1}(B_n)] = [v_{i_1+\dots+i_\alpha,i_1+\dots+i_\alpha+1}(A_{\{i_1,\dots,i_s\}})]$. □

Now let $\{i_1,\dots,i_s\}$ be some type. Fix a realization $\mathrm{sh}_{\{i_1,\dots,i_s\}} = \mathrm{sh}_{A_{\{i_1,\dots,i_s\}}}$. Define subvarieties of $\mathrm{sh}_{\{i_1,\dots,i_s\}}$:

$$N_j(\{i_1,\dots,i_s\}) = \overline{\left\{\exp\left(\sum_{l=0}^{n-1} z_l e_l + e^{(2)}_{n-j-1}z\right)\cdot[v_{j,j+1}(A_{\{i_1,\dots,i_s\}})], z_l, z\in\mathbb{C}\right\}}, \\ j = i_1, i_1+i_2,\dots, i_1+\dots+i_{s-1}.$$

From the lemma (3.2) we obtain the following corollary:

Corollary 3.2. *Let $s>1$. Then*

$$\mathrm{sh}_{\{i_1,\dots,i_s\}} \setminus \mathrm{G}\cdot[v_{\{i_1,\dots,i_s\}}] = \bigcup_{\alpha=1}^{s-1} N_{i_1+\dots+i_\alpha}(\{i_1,\dots,i_s\}).$$

Proof. Recall the surjective homomorphism $h_{\{i_1,\dots,i_s\}} : \mathrm{sh}^{(n)} \to \mathrm{sh}_{\{i_1,\dots,i_s\}}$. Note that the restriction

$$h_{\{i_1,\dots,i_s\}} : \mathrm{G}\cdot[v_{B_n}] \to \mathrm{G}\cdot[v_{\{i_1,\dots,i_s\}}]$$

is an isomorphism. In addition, lemma (3.2) gives us that

$$h_{\{i_1,\dots,i_s\}} N_j(\{1,\dots,1\}) = N_{i_1+\dots+i_\alpha}(\{i_1,\dots,i_s\}),$$
$$i_1+\dots+i_{\alpha-1} < j \le i_1+\dots+i_\alpha.$$

Corollary is proved. □

For the description of the varieties $N_j(\{i_1,\dots,i_s\})$ we need the following notation: let $\alpha_1,\dots,\alpha_{s_1},\beta_1,\dots,\beta_{s_2}$ be the natural numbers with

$$\alpha_1+\dots+\alpha_{s_1} = n-2, \quad \beta_1+\dots+\beta_{s_2} = n-j.$$

Consider the map

$$h_{\{\alpha_1,\dots,\alpha_{s_1}\}} \times h_{\{\beta_1,\dots,\beta_{s_2}\}} : \mathrm{sh}^{(n-2)} \times \mathrm{sh}^{(n-j)} \to \mathrm{sh}_{\{\alpha_1,\dots,\alpha_{s_1}\}} \times \mathrm{sh}_{\{\beta_1,\dots,\beta_{s_2}\}}.$$

Recall that $N_j(\{1,\dots,1\}) \hookrightarrow \mathrm{sh}^{(n-2)} \times \mathrm{sh}^{(n-j)}$. Define the variety

$$\mathrm{sh}_{\{\alpha_1,\dots,\alpha_{s_1}\}} \mathbin{\widetilde{\times}} \mathrm{sh}_{\{\beta_1,\dots,\beta_{s_2}\}} = h_{\{\alpha_1,\dots,\alpha_{s_1}\}} \times h_{\{\beta_1,\dots,\beta_{s_2}\}}(N_j(\{1,\dots,1\})).$$

The following proposition gives the description of the varieties $N_j(\{i_1,\dots,i_s\})$.

Proposition 3.1. *Let $A = A_{\{i_1,\dots,i_s\}}$.*

(1) $s=1$. *Then* $\mathrm{sh}_{\{n\}} = \mathrm{G}\cdot[v_A] \sqcup \mathrm{sh}_{\{n-2\}}$.

(2) $s>1$. *Then* $\mathrm{sh}_{\{i_1,\dots,i_s\}} = \mathrm{G}\cdot[v_A] \sqcup \bigcup_{\alpha=1}^{s-1} N_{i_1+\dots+i_\alpha}(\{i_1,\dots,i_s\})$ *and*

  (a) $i_\alpha \ge 2$. *Then*

  $$N_{i_1+\dots+i_\alpha}(\{i_1,\dots,i_s\}) \simeq \mathrm{sh}_{\{i_1,\dots,i_\alpha-2,\dots,i_s\}} \mathbin{\widetilde{\times}} \mathrm{sh}_{\{i_{\alpha+1},\dots,i_s\}}$$

  (if $i_\alpha = 2$ then $\{i_1,\dots,i_\alpha-2,\dots,i_s\} = \{i_1,\dots,i_{\alpha-1},i_{\alpha+1},\dots,i_s\}$).

  (b) $i_\alpha = 1$. *Then*

  $$N_{i_1+\dots+i_\alpha}(\{i_1,\dots,i_s\}) \simeq \mathrm{sh}_{\{i_1,\dots,i_{\alpha-1},i_{\alpha+1}-1,\dots,i_s\}} \mathbin{\widetilde{\times}} \mathrm{sh}_{\{i_{\alpha+1},\dots,i_s\}}.$$

Proof. Let $s=1$. Then $A=(2^n)$. Consider the map $h_{\{n\}} : \mathrm{sh}^{(n)} \to \mathrm{sh}_{\{n\}}$ and the vector $v'_A = e_{n-1}v_A \in M^A$. Then for any $j=1,\dots,n-1$ $h_{\{n\}}(v_{j,j+1}(B_n)) = v'_A$. The proof is similar to the one from the lemma (3.2). Thus

$$h_{\{n\}} N_j(\{1,\dots,1\}) = \overline{G\cdot[v'_A]}.$$

But $\phi_A(v'_A) = v(n-2)$. Hence $\overline{G\cdot[v'_A]} \simeq \mathrm{sh}_{\{n-2\}}$ and

$$\mathrm{sh}_{\{n\}} = \mathrm{G}\cdot[v_A] \sqcup \overline{G\cdot[v'_A]} \simeq \mathrm{G}\cdot[v_A] \sqcup \mathrm{sh}_{\{n-2\}}.$$

Now let $s>1$ and $i_\alpha \ge 2$. Because of the lemma (1.1) and the definition of the varieties $N_j(A)$ we obtain the embedding

$$N_{i_1+\dots+i_\alpha}(\{i_1,\dots,i_s\}) \hookrightarrow \mathrm{sh}_{\{i_1,\dots,i_\alpha-2,\dots,i_s\}} \times \mathrm{sh}_{\{i_{\alpha+1},\dots,i_s\}}.$$

At the same time the following diagram is commutative

$$\begin{array}{ccc}
N_{i_1+\dots+i_\alpha}(\{1,\dots,1\}) & \longrightarrow & \mathrm{sh}^{(n-2)} \times \mathrm{sh}^{(n-i_1-\dots-i_\alpha)} \\
{\scriptstyle h_{\{i_1,\dots,i_s\}}}\big\downarrow & & \big\downarrow{\scriptstyle h_{\{i_1,\dots,i_\alpha-2,\dots,i_s\}}\times h_{\{i_{\alpha+1},\dots,i_s\}}} \\
N_{i_1+\dots+i_\alpha}(A) & \longrightarrow & \mathrm{sh}_{\{i_1,\dots,i_\alpha-2,\dots,i_s\}} \times \mathrm{sh}_{\{i_{\alpha+1},\dots,i_s\}}.
\end{array}$$

This finishes the proof of the case (2a). The proof of the case (2b) is quite similar. □

As a corollary, we prove the theorem about the existence of the bundles:

Theorem 3.1. *Fix a type* $\{i_1, \dots, i_s\}$, $i_1 + \dots + i_s = n$. *For any* $t = 1, \dots, s-1$ *there is a* G*-equivariant bundle* $\mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{i_{t+1},\dots,i_s\}}$ *with a fiber* $\mathrm{sh}_{\{i_1,\dots,i_t\}}$, *sending* $[v_{\{i_1,\dots,i_s\}}]$ *to* $[v_{\{i_{t+1},\dots,i_s\}}]$.

Proof. Note that there is a natural G-equivariant surjective homomorphism

$$\Psi_t : \mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathrm{sh}_{\{i_{t+1},\dots,i_s\}}, \quad \Psi_t([v_{\{i_1,\dots,i_s\}}]) = [v_{\{i_{t+1},\dots,i_s\}}],$$

because

$$\begin{aligned}&\phi_{A_{\{i_1,\dots,i_s\}}}(v_{A_{\{i_1,\dots,i_s\}}}) = \\ &\quad = v(n) \otimes v(n-i_1) \otimes \dots \otimes v(n-i_1-\dots-i_t) \otimes \phi_{A_{\{i_{t+1},\dots,i_s\}}}(v_{A_{\{i_{t+1},\dots,i_s\}}}).\end{aligned}$$

We want to prove that for any $x \in \mathrm{sh}_{\{i_{t+1},\dots,i_s\}}$ the preimage $\Psi_t^{-1}(x)$ is isomorphic to $\mathrm{sh}_{\{i_1,\dots,i_t\}}$.

First prove that $\Psi_t^{-1}([v_{\{i_{t+1}\dots i_s\}}]) \simeq \mathrm{sh}_{\{i_1,\dots,i_t\}}$. Recall (see [FF1]) that

$$\mathbb{C}[e_{i_{t+1}+\dots+i_s}, \dots, e_{n-1}] \cdot v_{\{i_1,\dots,i_s\}} \simeq M^{(2^{i_1}\dots(t+1)^{i_t})}.$$

Thus we obtain

$$\begin{aligned}\Psi_t^{-1}([v_{\{i_{t+1}\dots i_s\}}]) &= \overline{\left\{\exp\left(\sum\nolimits_{j=i_{t+1}+\dots+i_s}^{n-1} e_j z_j\right) \cdot [v_{\{i_1,\dots,i_s\}}], z_j \in \mathbb{C}\right\}} \simeq \\ &\simeq \overline{\left\{\exp\left(\sum\nolimits_{j=0}^{i_1+\dots+i_t-1} e_j z_j\right) \cdot [v_{\{i_1,\dots,i_t\}}], z_j \in \mathbb{C}\right\}} \simeq \mathrm{sh}_{\{i_1,\dots,i_t\}}.\end{aligned}$$

In the same way one can prove that the preimage $\Psi_t^{-1}(x)$ is isomorphic to $\mathrm{sh}_{\{i_1,\dots,i_t\}}$ for any x from the orbit $\mathrm{G} \cdot [v_{\{i_{t+1},\dots,i_s\}}]$.

Recall that

$$\mathrm{sh}_{\{i_{t+1},\dots,i_s\}} = \mathrm{G} \cdot [v_{\{i_{t+1},\dots,i_s\}}] \bigsqcup \bigcup_{\alpha=1}^{s-t-1} N_{i_{t+1}+\dots+i_{t+\alpha}}(\{i_{t+1},\dots,i_s\}).$$

Pick α with $1 \le \alpha \le s-t-1$. Let $i_{t+\alpha} \ge 2$. Note that

$$\Psi_t^{-1} : \mathrm{sh}_{\{i_{t+1},\dots,i_\alpha-2,\dots,i_s\}} \,\widetilde{\times}\, \mathrm{sh}_{\{i_{\alpha+1},\dots,i_s\}} = \mathrm{sh}_{\{i_1,\dots,i_\alpha-2,\dots,i_s\}} \,\widetilde{\times}\, \mathrm{sh}_{\{i_{\alpha+1},\dots,i_s\}}$$

and for any

$$x \in \mathrm{sh}_{\{i_1,\dots,i_\alpha-2,\dots,i_s\}}, \quad y \in \mathrm{sh}_{\{i_{\alpha+1},\dots,i_s\}}$$

we have $\Psi_t(x \times y) = \Psi_t(x) \times y$. The assumption that

$$\Psi_t : \mathrm{sh}_{\{i_1,\dots,i_\alpha-2,\dots,i_s\}} \to \mathrm{sh}_{\{i_{t+1},\dots,i_\alpha-2,\dots,i_s\}}$$

is a bundle with a fiber $\mathrm{sh}_{\{i_1,\dots,i_t\}}$ gives us that for any

$$x \in N_{i_{t+1}+\dots+i_{t+\alpha}}(\{i_{t+1},\dots,i_s\}) \text{ with } i_{t+\alpha} \ge 2$$

the preimage $\Psi_t^{-1}(x)$ is isomorphic to $\mathrm{sh}_{\{i_1,\dots,i_t\}}$. The case of $i_{t+\alpha} = 1$ can be considered in the same way. Theorem is proved. ∎

Corollary 3.3. *Let* $P_{\{i_1,\dots,i_s\}}(q) = \sum_{j=0}^{2n} q^j \dim H_j(\mathrm{sh}_{\{i_1,\dots,i_s\}}, \mathbb{Z})$. *Then*

$$P_{\{i_1,\dots,i_s\}}(q) = \prod_{\alpha=1}^{s} \frac{(1-q^{2(i_\alpha+1)})}{(1-q^2)}.$$

Proof. Our corollary follows from the previous theorem and the statement that

$$P_{\{n\}}(q) = \frac{1-q^{2(n+1)}}{1-q^2}, \tag{6}$$

i.e. the odd homologies vanish and $\dim H_{2j}(\mathrm{sh}_{\{n\}}, \mathbb{Z}) = 1,\ j = 0, \dots, n$. Because of the proposition (3.1)

$$\mathrm{sh}_{\{n\}} = \mathrm{G} \cdot [v_{\{n\}}] \sqcup \mathrm{sh}_{\{n-2\}}. \tag{7}$$

But in [FF3] was proved that for any $A \in (\mathbb{N}\backslash)^n$ the orbit $\mathrm{G} \cdot [v_A] \hookrightarrow \mathrm{sh}_A$ is fibered over $\mathbb{P}^1$ with a fiber $\mathbb{C}^{n-1}$. Thus $\mathrm{G} \cdot [v_{\{n\}}]$ is the union of the two cells: $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$. Using (7) we obtain (6). □

Remark 3.1. *Consider the bundle* $\mathrm{G} \cdot [v_A] \to \mathbb{P}^1$. *It was proved in* [FF3] *that the transition functions of this bundle can be written in the following form. Let* $p \in \mathbb{P}^1 \setminus \{0, \infty\}$, x_0 *the coordinate of* p *in* $\mathbb{P}^1 \setminus \{\infty\}$, $y_0 = x_0^{-1}$ *the coordinate of* p *in* $\mathbb{P}^1 \setminus \{0\}$. *Denote by* $\{x_i\}, \{y_j\}$, $i, j = 1, \dots, n-1$ *the coordinates in the fiber* $\mathbb{C}^{n-1}$ *over the point* p *with respect to the trivializations on* $\mathbb{P}^1 \setminus \{\infty\}$ *and* $\mathbb{P}^1 \setminus \{0\}$. *Then we have an equality in the ring* $(\mathbb{C}[t]/t^n)$*:*

$$(x_0 + x_1 t + \ldots + x_{n-1} t^{n-1})(y_0 + y_1 t + \ldots + y_{n-1} t^{n-1}) = 1.$$

The Schubert varieties $\mathrm{sh}_{\{i_1, \dots, i_s\}}$ *can be considered as the* 2^{n-1} *ways of the compactification of the bundle over* $\mathbb{P}^1$ *with the fiber* $\mathbb{C}^{n-1}$ *and above transition functions.*

4. Schubert varieties as a generalized partial flag manifolds

Here we give a description of the varieties $\mathrm{sh}_{\{i_1, \dots, i_s\}}$ in terms of the special sequences of the subspaces of $\mathbb{C}[t] \oplus \mathbb{C}[t]$. We start with the case of $\mathrm{sh}_{\{n\}}$.

Lemma 4.1. *Let* $W_0 = \mathbb{C}^2 \otimes \mathbb{C}[t]$ *with a natural action of the operator* t *by multiplication. Define* $\mathrm{Fl}_{\{n\}}$ *as a variety of the subspaces* $W_1 \hookrightarrow W_0$ *with the following properties:*

$$1).\ \dim W_0/W_1 = n \qquad 2).\ tW_1 \hookrightarrow W_1 \qquad 3).\ W_1 \hookleftarrow t^n W_0. \tag{8}$$

Then $\mathrm{Fl}_{\{n\}} \simeq \mathrm{sh}_{\{n\}}$.

Proof. First note that because of the conditions 1) and 3) we can consider W_1 as a subspace of $\mathbb{C}^2 \otimes (\mathbb{C}[t]/t^n)$ of codimension n. Thus the group $\mathrm{G} = \mathrm{SL}_2(\mathbb{C}[t]/t^n)$ naturally acts on $\mathrm{Fl}_{\{n\}}$. Let v, u be the standard basis of the 2-dimensional $\mathfrak{sl}_2$ module, $hv = -v,\ hu = u,\ ev = u$. Denote $v_i = v \otimes t^i, u_i = u \otimes t^i, u_i, v_i \in W_0$. Let $V_{\{n\}} \in \mathrm{Fl}_{\{n\}}$ be the subspace with a basis $v_i, i = 0, \dots, n-1$. One can show that

$$\mathrm{Fl}_{\{n\}} = \overline{\mathrm{G} \cdot V_{\{n\}}}. \tag{9}$$

Consider a map

$$\mathrm{Fl}_{\{n\}} \to \bigwedge^n \left(\mathbb{C}^2 \otimes (\mathbb{C}[t]/t^n)\right), \quad W_1 \mapsto r_1 \wedge \ldots \wedge r_n, \tag{10}$$

where r_i is a basis of W_1. Surely (10) is a G-equivariant homomorphism. Thus, because of the formula (9) for the proof of the lemma it is enough to show that we have an isomorphism of $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^n)$ modules

$$\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^n) \cdot (v_0 \wedge \ldots \wedge v_{n-1}) \simeq M^{(2^n)}. \tag{11}$$

First we check that the defining relations from the right hand side of (11) are true in the left hand side. Recall (see [FF1]) that

$$M^{(2^n)} \simeq \mathbb{C}[e_0, \dots, e_{n-1}]/I,$$

where I is the ideal, generated by the following conditions

$$e^{(n)}(z)^i = (e_{n-1} + ze_{n-2} + \dots + z^{n-1}e_0)^i \div z^{n(i-1)}. \tag{12}$$

(The latter means that the first $n(i-1)-1$ coefficients of the series $e^{(n)}(z)^i$ vanish.) We will prove that $e^{(n)}(z)^i \cdot (v_0 \wedge \dots \wedge v_{n-1}) \div z^{n(i-1)}$. (One can easily check that the left hand side of (11) will not change after the replacement of $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^n)$ by $\mathbb{C}[e_0, \dots, e_{n-1}]$.)

By definition

$$\begin{aligned} (13) \quad & e^{(n)}(z)^i(v_0 \wedge \dots \wedge v_{n-1}) = \\ & = n! \sum_{0 \le \alpha_1 < \dots < \alpha_i \le n-1} v_0 \wedge \dots \wedge e^{(n)}(z)v_{\alpha_1} \wedge \dots \wedge e^{(n)}(z)v_{\alpha_i} \wedge \dots \wedge v_{n-1}. \end{aligned}$$

Let us show that $e^{(n)}(z)v_{\alpha_1} \wedge \dots \wedge e^{(n)}(z)v_{\alpha_i} \div z^{n(i-1)}$. In fact

$$\begin{aligned} (14) \quad & e^{(n)}(z)v_{\alpha_1} \wedge \dots \wedge e^{(n)}(z)v_{\alpha_i} = \\ & = \sum_{j_1=\alpha_1}^{n-1} u_{j_1} z^{n+\alpha_1-j_1-1} \wedge \sum_{j_2=\alpha_2}^{n-1} u_{j_2} z^{n+\alpha_2-j_2-1} \wedge \dots \wedge \sum_{j_i=\alpha_i}^{n-1} u_{j_i} z^{n+\alpha_i-j_i-1}. \end{aligned}$$

Note that

$$\begin{aligned} & \sum_{j_k=\alpha_k}^{n-1} u_{j_k} z^{n+\alpha_k-j_k-1} \wedge \sum_{j_{k+1}=\alpha_{k+1}}^{n-1} u_{j_{k+1}} z^{n+\alpha_{k+1}-j_{k+1}-1} = \\ & \qquad = \sum_{j_k=\alpha_k}^{\alpha_{k+1}-1} u_{j_k} z^{n+\alpha_k-j_k-1} \wedge \sum_{j_{k+1}=\alpha_{k+1}}^{n-1} u_{j_{k+1}} z^{n+\alpha_{k+1}-j_{k+1}-1}. \end{aligned}$$

Thus we can rewrite the formula (14) in the following way:

$$\begin{aligned} & e^{(n)}(z)v_{\alpha_1} \wedge \dots \wedge e^{(n)}(z)v_{\alpha_i} = \\ & = \sum_{j_1=\alpha_1}^{\alpha_2-1} u_{j_1} z^{n+\alpha_1-j_1-1} \wedge \sum_{j_2=\alpha_2}^{\alpha_3-1} u_{j_2} z^{n+\alpha_2-j_2-1} \wedge \dots \wedge \sum_{j_i=\alpha_i}^{n-1} u_{j_i} z^{n+\alpha_i-j_i-1} \div \\ & \qquad \div z^{n+\alpha_1-\alpha_2} z^{n+\alpha_2-\alpha_3} \dots z^{\alpha_i} = z^{n(i-1)+\alpha_1} \div z^{n(i-1)}. \end{aligned}$$

To finish the proof of the formula (11) we show that the following vectors are linearly independent ($k = 0, \dots, n$):

$$e_{i_1} \dots e_{i_k}(v_0 \wedge \dots \wedge v_{n-1}), \ i_\alpha \le n-k \ \ (1 \le \alpha \le k).$$

(That will be enough, because $\dim M^{(2^n)} = 2^n$.) Let

$$\sum_{0 \le i_1 \le \dots \le i_k \le n-k} \beta_{i_1,\dots,i_k} e_{i_1} \dots e_{i_k}(v_0 \wedge \dots \wedge v_{n-1}) = 0. \tag{15}$$

Pick such n-tuple $(i_1^0, \dots, i_k^0)$ that $\beta_{i_1^0,\dots,i_k^0} \ne 0$ and for any $(i_1, \dots, i_k)$ with a non-vanishing $\beta_{i_1,\dots,i_k}$ the following is true: there exists such $l \le k$ that $i_l > i_l^0$ and for

any $m < l$ $i_m = i_m^0$. Then the monomial

$$e_{i_1^0} v_0 \wedge \ldots \wedge e_{i_k^0} v_{k-1} \wedge v_k \wedge \ldots \wedge v_{n-1} = u_{i_1^0} \wedge u_{i_2^0+1} \wedge \ldots \wedge u_{i_k^0+k-1} \wedge v_k \wedge \ldots \wedge v_{n-1}$$

comes from $e_{i_1^0} \ldots e_{i_k^0}(v_0 \wedge \ldots \wedge v_{n-1})$ but not from any other monomial from the linear combination (15). Thus (11) is proved. $\square$

We will need the following lemma:

Lemma 4.2. *Let $i_1 + \ldots + i_s = n$. Then there is* G*-equivariant embedding*

$$\begin{aligned} \mathrm{sh}_{\{i_1,\ldots,i_s\}} &\hookrightarrow \mathrm{sh}_{\{n\}} \times \mathrm{sh}_{\{n-i_1\}} \times \ldots \times \mathrm{sh}_{\{n-i_1-\ldots-i_{s-1}\}}, \\ [v_{\{i_1,\ldots,i_s\}}] &\mapsto [v_{\{n\}}] \times [v_{\{n-i_1\}}] \times \ldots \times [v_{\{n-i_1-\ldots-i_{s-1}\}}]. \end{aligned} \tag{16}$$

Proof. Recall the embedding $M^{(2^{i_1}\ldots(s+1)^{i_s})} \hookrightarrow F^{\otimes s}$

$$v_{(2^{i_1}\ldots(s+1)^{i_s})} \mapsto v(n) \otimes v(n-i_1) \otimes \ldots \otimes v(n-i_1-\ldots-i_{s-1}).$$

Thus we obtain an embeddings

$$M^{(2^{i_1}\ldots(s+1)^{i_s})} \hookrightarrow M^{(2^n)} \otimes M^{(2^{n-i_1})} \otimes \ldots \otimes M^{(2^{n-i_1-\ldots-i_{s-1}})},$$
$$\mathrm{sh}_{\{i_1,\ldots,i_s\}} \hookrightarrow \mathrm{sh}_{\{n\}} \times \mathrm{sh}_{\{n-i_1\}} \times \ldots \times \mathrm{sh}_{\{n-i_1-\ldots-i_{s-1}\}}.$$

(Surely, the first embedding is a homomorphism of $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^n)$ modules and the second one is a G-equivariant homomorphism.) $\square$

Proposition 4.1. *Let $W_0 = \mathbb{C}^2 \otimes \mathbb{C}[t]$. Define the generalized partial flag manifold $\mathrm{Fl}_{\{i_1,\ldots,i_s\}}$ as the variety of the special sequences of the subspaces. Namely*

$$\begin{aligned} \mathrm{Fl}_{\{i_1,\ldots,i_s\}} = \{&W_0 \hookleftarrow W_1 \hookleftarrow \ldots \hookleftarrow W_s : \\ &1).\ tW_\alpha \hookrightarrow W_\alpha; \quad 2).\ \dim W_\alpha/W_{\alpha+1} = i_{s-\alpha}; \quad 3).\ W_{\alpha+1} \hookleftarrow t^{i_{s-\alpha}} W_\alpha\}. \end{aligned}$$

Then $\mathrm{Fl}_{\{i_1,\ldots,i_s\}} \simeq \mathrm{sh}_{\{i_1,\ldots,i_s\}}$.

Proof. Because of the condition 2) we know that $W_\alpha \hookleftarrow t^n W_0$. Thus we can consider W_α as a subspaces of $\mathbb{C}^2 \otimes (\mathbb{C}[t]/t^n)$. Define $V_{\{i_1,\ldots,i_s\}} \in \mathrm{Fl}_{\{i_1,\ldots,i_s\}}$ as follows:

$$V_{\{i_1,\ldots,i_s\}} = \{W_0 \hookleftarrow W_1 \hookleftarrow \ldots \hookleftarrow W_s\},$$
$$W_\alpha = \langle v_i, i = 0,\ldots,n-1;\ u_j, j = i_s + \ldots + i_{s-\alpha+1}, \ldots, n-1\rangle, 1 \le \alpha \le s-1,$$
$$W_s = \langle v_i, i = 0,\ldots,n-1\rangle.$$

Note that $\mathrm{Fl}_{\{i_1,\ldots,i_s\}} = \overline{\mathrm{G} \cdot V_{\{i_1,\ldots,i_s\}}}$. Define a map

$$\begin{aligned} \mathrm{Fl}_{\{i_1,\ldots,i_s\}} \to &\bigwedge^{n+i_1+\ldots+i_{s-1}} \left(\mathbb{C}^2 \otimes (\mathbb{C}[t]/t^n)\right) \oplus \\ &\oplus \bigwedge^{n+i_1+\ldots+i_{s-2}} \left(\mathbb{C}^2 \otimes (\mathbb{C}[t]/t^n)\right) \oplus \ldots \oplus \bigwedge^{n} \left(\mathbb{C}^2 \otimes (\mathbb{C}[t]/t^n)\right), \\ &\{W_0 \hookleftarrow W_1 \hookleftarrow \ldots \hookleftarrow W_s\} \mapsto \bigoplus_{\alpha=1}^{s} r_1^\alpha \wedge \ldots \wedge r^\alpha_{n+i_1+\ldots+i_{s-\alpha}}, \end{aligned} \tag{17}$$

where r_j^α form a basis of W_α. Consider an embedding $\chi_\alpha, \alpha = 1,\ldots,s$:

$$\chi_\alpha : \bigwedge^{i_s+\ldots+i_{s-\alpha+1}} (\mathbb{C}^2 \otimes (\mathbb{C}[t]/t^{i_s+\ldots+i_{s-\alpha+1}})) \hookrightarrow \bigwedge^{n+i_1+\ldots+i_{s-\alpha}} (\mathbb{C}[t]/t^n),$$
$$\chi_\alpha(w) = w \wedge v_{i_s+\ldots+i_{s-\alpha+1}} \wedge \ldots \wedge v_{n-1} \wedge u_{i_s+\ldots+i_{s-\alpha+1}} \wedge \ldots \wedge u_{n-1}.$$

One can show that the image of the map (17) belongs to the direct sum of the images $\bigoplus_{\alpha=1}^{s} \mathrm{im}(\chi_\alpha)$. In addition, $V_{\{i_1,\dots,i_s\}} \mapsto V_{\{i_s\}} \times V_{\{i_s+i_{s-1}\}} \times \dots \times V_{\{n\}}$. Thus we obtain a map

$$\mathrm{Fl}_{\{i_1,\dots,i_s\}} \hookrightarrow \mathrm{Fl}_{\{i_s\}} \times \mathrm{Fl}_{\{i_s+i_{s-1}\}} \times \dots \times \mathrm{Fl}_{\{n\}}.$$

Because of the lemmas (4.1) and (4.2) we obtain the G-equivariant isomorphism $\mathrm{Fl}_{\{i_1,\dots,i_s\}} \simeq \mathrm{sh}_{\{i_1,\dots,i_s\}}$. □

5. Line bundles on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$

Fix the set of the curves $C_j \hookrightarrow \mathrm{sh}_{\{i_1,\dots,i_s\}}, j = 0,\dots,n-1$,

$$C_j = \overline{\{\exp(ze_j)\cdot[v_{\{i_1,\dots,i_s\}}],\ z \in \mathbb{C}\}} \simeq \mathbb{P}^1.$$

Let $\mathcal{E}$ be the line bundle on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$. We write $\mathcal{E} = \mathcal{O}(a_1,\dots,a_n) = \mathcal{O}(A)$ ($a_i \in \mathbb{Z}$) if

$$\mathcal{E}|_{C_j} = \mathcal{O}(a_1 + \dots + a_{n-j}).$$

In [FF3] was shown that the bundle on $\mathrm{sh}^{(n)}$ is uniquely determined by the numbers a_i. The same statement is true for the general $\mathrm{sh}_{\{i_1,\dots,i_s\}}$. In the case of $\mathrm{sh}^{(n)}$ for any $\{a_i\}$ there exists a bundle $\mathcal{O}(a_1,\dots,a_n)$. The general case is considered in the proposition (5.1).

Lemma 5.1. *Consider the homomorphism $h_{\{i_1,\dots,i_s\}} : \mathrm{sh}^{(n)} \to \mathrm{sh}_{\{i_1,\dots,i_s\}}$. Recall the subvariety $N_{n-1}(\{1,\dots,1\}) \simeq \mathrm{sh}^{(n-2)} \times \mathbb{P}^1$ of $\mathrm{sh}^{(n)}$. Denote by L the following projective line:*

$$L = [v_{\{\underbrace{1,\dots,1}_{n-2}\}}] \times \mathbb{P}^1 \hookrightarrow N_{n-1}(\underbrace{1,\dots,1}_{n}) \hookrightarrow \mathrm{sh}^{(n)}. \tag{18}$$

If $i_s > 1$ then $h_{\{i_1,\dots,i_s\}}$ maps L to the point.

Proof. From the results of [FF3] follows that

$$L = \mathrm{SL}_2^{(n)} \cdot [v(n-2) \otimes v(n-3) \otimes \dots \otimes v(0) \otimes v(1)], \tag{19}$$

where $\mathrm{SL}_2^{(n)}$ stands for the SL_2 acting on the n-th (last) factor of the tensor power $F^{\otimes n}$.

Recall (see lemma (3.1), corollary (3.1)) that the map $h_{\{i_1,\dots,i_s\}}$ can be regarded as a part of the following commutative diagram (the horizontal arrows come from the lemma (4.2)):

$$\begin{array}{ccc}
\mathrm{sh}^{(n)} & \longrightarrow & \mathrm{sh}_{\{n\}} \times \mathrm{sh}_{\{n-1\}} \times \dots \times \mathrm{sh}_{\{1\}} \\
{\scriptstyle h_{\{i_1,\dots,i_s\}}}\downarrow & & \downarrow{\scriptstyle P} \\
\mathrm{sh}_{\{i_1,\dots,i_s\}} & \longrightarrow & \mathrm{sh}_{\{n\}} \times \mathrm{sh}_{\{n-i_1\}} \times \dots \times \mathrm{sh}_{\{i_s\}}
\end{array}$$

where P is a map of "forgetting" of some factors. In the case $i_s > 1$ the last, n-th factor is one to "forget". Thus because of the formula (19) $h_{\{i_1,\dots,i_s\}}$ maps L to the point. □

Proposition 5.1. *Let $A = (a_1,\dots,a_n) \in \mathbb{Z}^n$ be of the type $\{j_1,\dots,j_{s_1}\}$. Then the bundle $\mathcal{O}(A)$ on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$ really exists if and only if $\{j_1,\dots,j_{s_1}\} \le \{i_1,\dots,i_s\}$.*

Proof. First note that for any $A = (a_1, \dots, a_n), a_i > 0$ of the type less or equal to $\{i_1, \dots, i_s\}$ there exists homomorphism $\imath_A : \mathrm{sh}_{\{i_1,\dots,i_s\}} \to \mathbb{P}(M^A)$. In fact, if $a_{i_1+\dots+i_\alpha} \neq a_{i_1+\dots+i_\alpha+1}$ for all α, then $\imath_A$ is an embedding coming from the realization $\mathrm{sh}_{\{i_1,\dots,i_s\}} = \mathrm{sh}_A$. If there exists α with $a_{i_1+\dots+i_\alpha} = a_{i_1+\dots+i_\alpha+1}$, then $\imath_A$ is a composition of the above embedding and a homomorphisms from the lemma (3.1). It was shown in [FF3] that

$$\imath_A^* \mathcal{O}(1) = \mathcal{O}(a_1 - 1, \dots, a_n - 1).$$

Thus for any A with $\{j_1, \dots, j_{s_1}\} \le \{i_1, \dots, i_s\}$ and $a_i \ge 0$ the line bundle $\mathcal{O}(A)$ really exists on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$. The case of an arbitrary A is an immediate consequence, because any A of the type $\{j_1, \dots, j_{s_1}\}$ can be represented as $A = B - C$ ($a_i = b_i - c_i$), where $B, C \in \mathbb{N}^n$.

Now ewe need to prove that the bundle $\mathcal{O}(A)$ really exists on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$ only if the type of A is less or equal to $\{i_1, \dots, i_s\}$. Recall that

$$\overline{\left\{\exp\left(\sum_{j=1}^{n-1} z_j e_j\right) \cdot [v_{\{i_1,\dots,i_s\}}],\ z_j \in \mathbb{C}\right\}} \simeq \mathrm{sh}_{\{i_1,\dots,i_{s-1},i_s-1\}}.$$

The restriction of $\mathcal{E} = \mathcal{O}(a_1, \dots, a_n)$ to this subvariety equals to $\mathcal{O}(a_1, \dots, a_{n-1})$. Using the induction assumption we know that $\{j_1, \dots, j_{s_1} - 1\} \le \{i_1, \dots, i_s - 1\}$. Note that in the case $i_s = 1$ we obtain that the type of A is less or equal to $\{i_1, \dots, i_s\}$.

Let $i_s > 1$. We want to prove that $j_{s_1} > 1$. Consider the variety $\mathrm{sh}^{(n)}$ and the line $L \hookrightarrow \mathrm{sh}^{(n)}$ (see lemma (5.1)). It was shown in [FF3] that the restriction of the bundle $\mathcal{O}(A)$ on $\mathrm{sh}^{(n)}$ to this line equals to $\mathcal{O}(a_n - a_{n-1})$. Using the lemma (5.1) we obtain that if $\mathcal{O}(A)$ really exists on $\mathrm{sh}_{\{i_1,\dots,i_s\}}$ then $a_n - a_{n-1} = 0$. Thus $j_{s_1} > 1$. Theorem is completely proved. □

Corollary 5.1. *Let $A \in \mathbb{N}^n$ be of the type less or equal to $\{i_1, \dots, i_s\}$. Then*

$$H^0(\mathcal{O}(a_1 - 1, \dots, a_n - 1), \mathrm{sh}_{\{i_1,\dots,i_s\}}) \simeq (M^A)^*.$$

Proof. It was proved in [FF3] that

$$H^0(\mathcal{O}(a_1 - 1, \dots, a_n - 1), \mathrm{sh}^{(n)}) \simeq (M^A)^*.$$

The statement of the corollary follows from the fact that

$$h^*_{\{i_1,\dots,i_s\}} \mathcal{O}(a_1 - 1, \dots, a_n - 1) = \mathcal{O}(a_1 - 1, \dots, a_n - 1)$$

and the induced map of the sections is an isomorphism. □

6. Infinite-dimensional consequences

Recall that in [FF3] for any $A = (a_1 \le \dots \le a_n) \in \mathbb{N}^n$ the infinite-dimensional variety Gr_A was defined as an inductive limit of the Schubert varieties. Namely, let

$$A^{(i)} = (a_1, \dots, a_n, \underbrace{a_n, \dots, a_n}_{2i}).$$

Then we have an embedding $\mathrm{sh}_{(A^{(i)})} \hookrightarrow \mathrm{sh}_{(A^{(i+1)})}$ and $Gr_A = \lim_{i\to\infty} \mathrm{sh}_{(A^{(i)})}$. Because of the theorem (2.1) we have a collection of varieties $Gr_{\{i_1,\dots,i_s\}}$:

$$Gr_{\{i_1,\dots,i_s\}} = \lim_{i\to\infty} \mathrm{sh}_{\{i_1,\dots,i_s+2i\}}.$$

By definition we have an action of the group $\mathrm{SL}_2(\mathbb{C}[t])$ on $Gr_{\{i_1,\dots,i_s\}}$. It was shown in [FF2, FF3] that in fact we have an action of the group $\widehat{\mathrm{SL}_2}$ (the reason is that the space $\lim_{i\to\infty} M^{A^{(i)}}$ is not only $\mathfrak{sl}_2\otimes\mathbb{C}[t]$ module, but also has a structure of a representation of the Lie algebra $\widehat{\mathfrak{sl}_2}$).

Lemma 6.1. *Let* $\jmath : \mathrm{sh}_{\{i_1,\dots,i_s\}} \hookrightarrow \mathrm{sh}_{\{i_1,\dots,i_s+2\}}$, $B=(b_1,\dots,b_n)\in\mathbb{Z}^n$ *the element of the type* $\{i_1,\dots,i_s\}$. *Then* $\jmath^*\mathcal{O}(B^{(1)})=\mathcal{O}(B)$.

Proof. It is enough to prove our lemma for $B\in(\mathbb{N}\setminus 1)^n$. In this case the embedding $\jmath$ is the restriction of the embedding of the projective spaces $\widehat{\jmath} : \mathbb{P}(M^B)\hookrightarrow\mathbb{P}(M^{B^{(1)}})$, constructed in [FF3] (we use the embeddings $\imath_B : \mathrm{sh}_{\{i_1,\dots,i_s\}}\hookrightarrow\mathbb{P}(M^B)$ and $\imath_{B^{(1)}} : \mathrm{sh}_{\{i_1,\dots,i_s+2\}}\hookrightarrow\mathbb{P}(M^{B^{(1)}})$). Thus $\imath_{B^{(1)}}\jmath=\widehat{\jmath}\imath_B$. We obtain

$$\jmath^*\mathcal{O}(B^{(1)})=\jmath^*\imath^*_{B^{(1)}}\mathcal{O}_{\mathbb{P}(M^{B^{(1)}})}(1)=\imath^*_B\widehat{\jmath}^*\mathcal{O}_{\mathbb{P}(M^{B^{(1)}})}(1)=\imath^*_B\mathcal{O}_{\mathbb{P}(M^B)}(1)=\mathcal{O}(B).$$

Lemma is proved. $\square$

Now let $\mathbf{E}$ be a line bundle on $Gr_{\{i_1,\dots,i_s\}}$. We write

$$\mathbf{E}=\mathcal{O}(B^{(\infty)}) \ \text{ if } \ \mathbf{E}|_{\mathrm{sh}_{\{i_1,\dots,i_s+2i\}}}=\mathcal{O}(B^{(i)}).$$

Let $B\in\mathbb{N}^n$. Consider the projective limit of the spaces of sections

$$H^0(\mathcal{O}(B^{(\infty)}),Gr_{\{i_1,\dots,i_s\}})=\varprojlim_i H^0(\mathcal{O}(B^{(i)}),\mathrm{sh}_{\{i_1,\dots,i_s+2i\}})=$$
$$=\varprojlim_i\left(M^{(b_1+1,\dots,b_n+1,(b_n+1)^{2i})}\right)^*.$$

It was proved in [FF2] that we have an isomorphism of $\widehat{\mathfrak{sl}_2}$ modules:

$$\varinjlim_i M^{(b_1+1,\dots,b_n+1,(b_n+1)^{2i})}\simeq\bigoplus_{j=0}^{b_n}c_{j;b_1,\dots,b_n}L_{j,b_n},$$

where L_{j,b_n} are level b_n irreducible $\widehat{\mathfrak{sl}_2}$ modules and the numbers $c_{j;b_1,\dots,b_n}$ are defined by the following equation in the Verlinde algebra of the level b_n+1, associated with the Lie algebra $\mathfrak{sl}_2$:

$$[b_1]\cdot\ldots\cdot[b_n]=\sum_{j=0}^{b_n}c_{j;b_1,\dots,b_n}[j]$$

(here the notation $[j]$ stands for the element of the Verlinde algebra, corresponding to the $[j+1]$-dimensional irreducible representation of $\mathfrak{sl}_2$). Thus we obtain the following proposition:

Proposition 6.1. *Let* $B\in\mathbb{N}^n$. *Then we have an isomorphism of* $\widehat{\mathfrak{sl}_2}$ *modules*

$$H^0(\mathcal{O}(B^{(\infty)}),Gr_{\{i_1,\dots,i_s\}})\simeq\bigoplus_{j=0}^{b_n}c_{j;b_1,\dots,b_n}L^*_{j,b_n}.$$

7. Discussion of the singularities of the Schubert varieties

Recall that in [FF3] was shown that $\mathrm{sh}^{(n)}$ is a smooth variety.

Lemma 7.1. *Let $i_1+\ldots+i_s=n$ and $s\neq n$. Then $\mathrm{sh}_{\{i_1,\ldots,i_s\}}$ is a singular variety.*

Proof. Recall (see [FF3]) that there is a bundle

$$\widetilde{\pi}:\mathrm{G}\cdot[v_{\{i_1,\ldots,i_s\}}]\to \mathrm{SL}_2\cdot[v_{\{i_1,\ldots,i_s\}}]\simeq\mathbb{P}^1$$

with a fiber $\mathbb{C}^{n-1}$. It is easy to show that the closure of the fiber $\widetilde{\pi}(x)^{-1}$ is isomorphic to $\mathrm{sh}_{\{i_1,\ldots,i_{s-1},i_s-1\}}$ for any $x\in\mathbb{P}^1$. Note that because $\widetilde{\pi}$ is SL_2-equivariant homomorphism, $\mathrm{sh}_{\{i_1,\ldots,i_s\}}$ is smooth if and only if

1. $\overline{\widetilde{\pi}(x)^{-1}}$ is smooth for any $x\in\mathbb{P}^1$,
2. $\overline{\widetilde{\pi}(x)^{-1}}\cap\overline{\widetilde{\pi}(y)^{-1}}=\emptyset$ if $x\neq y$.

Suppose that we have already proved our lemma for $\sum i_\alpha<n$. Then the first condition means that $i_1=\ldots=i_{s-1}=1, i_s\le 2$. In [FF3] was proved that the condition (2) holds for $i_s=1$. In the same way one can prove that otherwise the closures of the fibers of $\widetilde{\pi}$ intersect. Thus $i_s=1$ and lemma is proved. $\square$

It is interesting to describe the variety of the singular points of $\mathrm{sh}_{\{i_1,\ldots,i_s\}}$ in the general case. In the next proposition we consider the case of $\mathrm{sh}_{\{n\}}$.

Proposition 7.1. *Recall (see proposition (3.1)) that $\mathrm{sh}_{\{n\}}=\mathrm{G}\cdot[v_{\{n\}}]\sqcup N$ and $N\simeq\mathrm{sh}_{\{n-2\}}$.*

1. *N is a variety of the singular points of $\mathrm{sh}_{\{n\}}$.*
2. *There exists a line bundle $\mathcal{E}$ on $\mathrm{sh}_{\{n\}}$ and $s_1,s_2\in H^0(\mathcal{E},\mathrm{sh}_{\{n\}})$ such that $N=\{s_1=0\}\cap\{s_2=0\}$.*

Proof. For the proof of the first part of the proposition it is enough to show that for any $x\in\mathbb{P}^1$ variety N is contained in the closure $\overline{\widetilde{\pi}^{-1}(x)}$. First let us show the latter for the point $x=[v_{\{n\}}]$ (recall that we have a realization $\mathbb{P}^1=\mathrm{SL}_2\cdot[v_{\{n\}}]$). Note that the case of an arbitrary x is an immediate consequence, because $\widetilde{\pi}$ is an SL_2-equivariant map.

Recall that $\mathrm{sh}_{\{n\}}=\mathrm{sh}_{(2^n)}$. From the proof of the proposition (3.1) in the case of $\mathrm{sh}_{\{n\}}$ follows that

$$\lim_{z\to\infty}\exp(e_{n-1}z)[v_{\{n\}}]=[e_{n-1}v_{(2^n)}]=[v_{\{n-2\}}]\in\mathrm{sh}_{\{n-2\}}\simeq N$$

(recall that $e_{n-1}^2v_{(2^n)}=0$). Thus because of

$$N=\overline{\mathrm{SL}_2(\mathbb{C}[t]/t^{n-2})\cdot[e_{n-1}v_{(2^n)}]}$$

we obtain $N\hookrightarrow\overline{\widetilde{\pi}^{-1}([v_{\{n\}}])}$. The first part of the proposition is proved.

Recall that $\mathrm{Pic}(\mathrm{sh}_{\{n\}})\simeq\mathbb{Z}$. Let $\mathcal{E}=\mathcal{O}(1,\ldots,1)$ be the generator of this group. We have an isomorphism (see corollary (5.1)):

$$H^0(\mathcal{E},\mathrm{sh}_{\{n\}})\simeq\left(M^{(2^n)}\right)^*. \tag{20}$$

Fix some basis $\{b_i\}_{i=1}^{2^n}$ of $M^{(2^n)}$ which is homogeneous with respect to the h_0-grading and $b_1=v_{(2^n)},b_{2^n}=u_{(2^n)}$. Let $s_1,s_2\in H^0(\mathcal{E},\mathrm{sh}_{\{n\}})$, $s_1=v^*_{\{n\}},s_2=u^*_{\{n\}}$ (thus s_1,s_2 are the elements of the dual basis b_i^*). Then

$$\{s_1=0\}=\overline{\widetilde{\pi}^{-1}([u_{\{n\}}])},\quad \{s_2=0\}=\overline{\widetilde{\pi}^{-1}([v_{\{n\}}])}. \tag{21}$$

Let us prove that the first equality really holds (the proof for the second one is quite similar). In fact,

(22)

$$\mathrm{sh}_{\{n\}} = \left\{ \exp\left(\sum_{j=0}^{n-1} e_j z_j\right) \cdot [v_{\{n\}}], z_j \in \mathbb{C} \right\} \bigsqcup \overline{\left\{ \exp\left(\sum_{j=1}^{n-1} z_j f_j\right) \cdot [u_{\{n\}}], z_j \in \mathbb{C} \right\}}.$$

Rewrite (22) as $\mathrm{sh}_{\{n\}} = R_1 \sqcup R_2$. Recall the embedding $\imath_{(2^n)} : \mathrm{sh}_{\{n\}} \hookrightarrow \mathbb{P}(M^{(2^n)})$ and the equality $\imath^*_{(2^n)} \mathcal{O}(1) = \mathcal{E}$. Note that (20) means that the restriction map $H^0(\mathcal{O}(1), \mathbb{P}(M^{(2^n)})) \to H^0(\mathcal{E}, \mathrm{sh}_{\{n\}})$ is an isomorphism. Thus $\{s_1 = 0\} \cap R_1 = \emptyset$. In addition $R_2 \hookrightarrow \mathbb{P}(\langle b_i, i > 1 \rangle)$ and so $s_1|_{R_2} = 0$. But $R_2 = \overline{\widetilde{\pi}^{-1}([u_{\{n\}}])}$. We have proved (21) and hence our proposition is completely proved. $\square$

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